5.5 Integration by Substitution

Goals:
1. Recognize an integrand that is the derivative of a composite function.
2. Generalize the Basic Integration Rules to include composite functions.
3. Use substitution to simplify the process of integration of composite functions.

Study 5.5, # 1-5, 9, 13-27, 35, 39, 49-59, 63, 69, 71, 81

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Homework

2.4 Chain Rule: Derivative of Composite Functions

Revisit, from 2.4:

Chain Rule

If:
1) $y = f(u)$ is a differentiable function of u, and
2) $u=g(x)$ is a differentiable function of x,

then

$y = f(g(x))$ is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

or

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

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5.5 Integration by Substitution

Use the Chain Rule

G: $y = (x^2 + 2)^{10}$ F: dy/dx

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5.5 Integration by Substitution

Use the Chain Rule

G: $y = (x^2 + 2)^{10}$ F: dy/dx

$dy/dx = 10\,(x^2 + 2)^9\,(2x)$

$dy/dx = 20x\,(x^2 + 2)^9$

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5.5 Integration by Substitution

Remember the Chain Rule

$y = (x^2 + 2)^{10}$ $dy/dx = 20x\,(x^2 + 2)^9$

Complete the table

y	dy/dx
$y = (2x + 1)^3$	
$y = \sin(x^2)$	
	$dy/dx = 2\cos(2x+1)$
$y = \tan(x^2)$	
	$dy/dx = 3x^2 \sec^2(x^3)$

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5.5 Integration by Substitution

Remember the Chain Rule

$y = (x^2 + 2)^{10}$ $dy/dx = 20x\,(x^2 + 2)^9$

Complete the table

y	dy/dx
$y = (2x + 1)^3$	$dy/dx = 6\,(2x+1)^2$
$y = \sin(x^2)$	$dy/dx = 2x\cos(x^2)$
	$dy/dx = 2\cos(2x+1)$
$y = \tan(x^2)$	$dy/dx = 2x \sec^2(x^2)$
	$dy/dx = 3x^2 \sec^2(x^3)$

1. All derivatives here use the Chain Rule to find the derivative of composite functions.

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5.5 Integration by Substitution

Remember the Chain Rule

$y = (x^2 + 2)^{10}$ $dy/dx = 20x\,(x^2 + 2)^9$

Complete the table

y	dy/dx
$y = (2x + 1)^3$	$dy/dx = 6\,(2x+1)^2$
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	$dy/dx = 2\,\cos(2x+1)$
$y = \tan(x^2)$	$dy/dx = 2x\,\sec^2(x^2)$
	$dy/dx = 3x^2\,\sec^2(x^3)$

——— Derivative of the primary function.
═══ Derivative of the nested function.

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5.5 Integration by Substitution

Remember the Chain Rule

Complete the table

$y = (2x + 1)^3$	$dy/dx = 6\,(2x+1)^2$
$y = \sin(x^2)$	$dy/dx = 2x\,\cos(x^2)$
$y = \sin(2x+1) + c$	$dy/dx = 2\,\cos(2x+1)$
$y = \tan(x^2)$	$dy/dx = 2x\,\sec^2(x^2)$
$y = \tan(x^3) + c$	$dy/dx = 3x^2\,\sec^2(x^3)$

——— Derivative of the primary function.
═══ Derivative of the nested function.

2. To find the integrals of functions that are the derivatives of composite functions, **the integrand requires the presence of the derivative of the nested function as a factor.** This is the reverse process of the Chain Rule.

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5.5 Integration by Substitution

$\int (x^2 - 9)^3 (2x)dx$

u = ______

du = ________

$\int 2x (x^2 - 9)^3 dx$

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5.5 Integration by Substitution

$\int (x^2 - 9)^3 (2x)dx$

$u = x^2 - 9$

$du = 2x\, dx$

u substitution

$\int u^3 du$

$\frac{u^4}{4} + c$

$\frac{(x^2 - 9)^4}{4} + c$

$\int 2x (x^2 - 9)^3 dx$

same as original

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5.5 Integration by Substitution

$$\int x^2\sqrt{x^3+1}\,dx$$

u = ______

du = ________

5.5 Integration by Substitution

$$\int x^2\sqrt{x^3+1}\,dx$$

$u = x^3+1$

$du = 3x^2\,dx$

$$\frac{1}{3}\int (x^3+1)^{1/2}\;3x^2dx$$

$$\frac{1}{3}\int u^{1/2}\,du$$

$$\frac{1}{3}\,\frac{u^{3/2}}{\frac{3}{2}} + c = \frac{2}{9}(x^3+1)^{3/2} + c$$

5.5 Integration by Substitution

$$\int 2 \sec 2x \tan 2x \, dx$$

u = ______

du = ________

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5.5 Integration by Substitution

$$\int 2 \sec 2x \tan 2x \, dx$$

$u = 2x$

$du = 2\,dx$

$$\int \sec u \tan u \, du$$

$$\sec u + c$$

$$\sec 2x + c$$

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5.5 Integration by Substitution

$$\int \sec 2x \tan 2x \, dx$$

u = ______

du = ________

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5.5 Integration by Substitution

$$\int \sec 2x \tan 2x \, dx$$

u = 2x

du = 2 dx

$$\tfrac{1}{2} \int \sec 2x \tan 2x \; 2dx$$

$$\tfrac{1}{2} \int \sec u \tan u \; du$$

$$\tfrac{1}{2} \sec u + c$$

$$\tfrac{1}{2} \sec 2x + c$$

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5.5 Integration by Substitution

$\int \sin x \cos x \, dx$

u = ______

du = ________

5.5 Integration by Substitution

$\int \sin x \cos x \, dx$

u = sin x

du = cos x dx

$\int u \, du$

$\frac{u^2}{2} + c$

$\frac{1}{2} (\sin x)^2 + c$

5.5 Integration by Substitution

B

$\int \sin x \cos x \, dx$

u = cos x

du = ________

5.5 Integration by Substitution

$\int \sin x \cos x \, dx$

u = cos x

du = - sin x dx

$-\int -\sin x \cos x \, dx$

$-\int u \, du$

$-\frac{u^2}{2} + c$

$-\frac{1}{2}(\cos x)^2 + c$

5.5 Integration by Substitution

$\int \sin x \cos x \, dx$

$\frac{1}{2} \int 2 \sin x \cos x \, dx$

$\frac{1}{2} \int \sin 2x \, dx$

Trig Identity:
sin 2x = 2 sin x cos x

u = 2x

du = ________

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5.5 Integration by Substitution

$\int \sin x \cos x \, dx$

$\frac{1}{2} \int 2 \sin x \cos x \, dx$

$\frac{1}{2} \int \sin 2x \, dx$

$\frac{1}{2} \frac{1}{2} \int \sin 2x \, 2dx$

$\frac{1}{4} \int \sin u \, du$

$-\frac{1}{4} \cos u + c$

$-\frac{1}{4} \cos 2x + c$

Trig Identity:
sin 2x = 2 sin x cos x

u = 2x

du = 2 dx

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5.5 Integration by Substitution

$\int \sin x \cos x \, dx$

Trig Identity:
$\sin 2x = 2 \sin x \cos x$

3 solutions look different, but they are equivalent, different only by constants

$\frac{1}{2} (\sin x)^2 + c_1$

$-\frac{1}{2} (\cos x)^2 + c_2$

$-\frac{1}{4} \cos 2x + c_3$

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5.5 Integration by Substitution

Rules of Integration

$\int k \, dx = kx + c$

$\int u^n \, du = \frac{u^{n+1}}{n+1} + c$

$\int e^u \, du = e^u + c$

$\int \frac{du}{u} = \ln |u| + c$

$\int \sin u \, du = -\cos u + c$

$\int \cos u \, du = \sin u + c$

$\int \sec u \tan u \, du = \sec u + c$

$\int \sec^2 u \, du = \tan u + c$

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5.5 Integration by Substitution

Rules Integration

$$\int (x+1)\sqrt{2 - x}\, dx$$

u = ______

du = ________

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5.5 Integration by Substitution

Rules Integration

$$\int (x+1)\sqrt{2 - x}\, dx$$

$$-\int (x+1)(2 - x)^{½} (-)\, dx$$

$u = 2 - x$

$du = - dx$

Note: **x+1 is neither part of *u* nor *du***, but we need to convert all x to u before continuing. The conversion of x+1 to a function of u must be consistent with u=2-x:

u= 2-x; so x= 2-u, and x+1= 2-u+1 = **3-u**

$$-\int (3-u)(u)^{½}\, du = - \int (3u^{½} - u^{3/2})\, du$$

$$= - \frac{3u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + c$$

$$= - 2(2-x)^{3/2} + \frac{2}{5}(2-x)^{5/2} + c$$

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5.5 Integration by Substitution

Rules Integration

8

$$\int x^2 e^{x^3} dx$$

u = ______

du = ______

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5.5 Integration by Substitution

Rules Integration

$$\int x^2 e^{x^3} dx$$

$u = x^3$

$du = 3x^2 \, dx$

$$1/3 \int e^{x^3} \, 3x^2 \, dx$$

$$1/3 \int e^u \, du$$

$$\frac{e^u}{3} + c = \frac{e^{x^3}}{3} + c$$

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5.5 Integration by Substitution

Rules Integration

32

$$\int \frac{x}{x^2 + 4}\, dx$$

u = ______

du = ______

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5.5 Integration by Substitution

Rules Integration

32

$$\int \frac{x}{x^2 + 4}\, dx$$

$u = x^2 + 4$

$du = 2x\, dx$

$$\tfrac{1}{2}\int \frac{2x}{x^2 + 4}\, dx$$

$$\tfrac{1}{2}\int \frac{du}{u}$$

$$\tfrac{1}{2} \ln | u | + c$$

$$\tfrac{1}{2} \ln | x^2 + 4 | + c = \tfrac{1}{2} \ln (x^2 + 4) + c$$

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5.5 Integration by Substitution
60

Rules Integration

$$\int_0^1 x\, e^{-x^2}\, dx$$

u = ______

du = ______

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5.5 Integration by Substitution
60

$u = -x^2$

du = -2x dx

$$\int_0^1 x\, e^{-x^2}\, dx$$

Limit	x	$u=-x^2$
LL	0	0
UL	1	-1

$$-\tfrac{1}{2}\int_0^1 e^{-x^2}(-2x)\, dx$$

$$-\tfrac{1}{2}\int_0^{-1} e^u\, du$$

$$\tfrac{1}{2}\int_{-1}^0 e^u\, du = \tfrac{1}{2}\, e^u \Big]_{-1}^{0} = \tfrac{1}{2}\,[e^0 - e^{-1}]$$

$$= \tfrac{1}{2}\,[1 - 1/e] = \frac{e-1}{2e}$$

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5.5 Integration by Substitution

Rules Integration

82 G: A bacteria population starts with 400 bacteria and grows at the rate in bacteria/hr as noted below.

$$r(t) = (450.268)e^{1.125673t}$$

F: How many bacteria will there be after 3 hours?

Amt after 3 hours = original amt ± net change

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5.5 Integration by Substitution

82 G: A bacteria population starts with 400 bacteria and grows at the rate in bacteria/hr as noted below.

$$r(t) = (450.268)e^{1.125673t}$$

F: How many bacteria will there be after 3 hours?

Amt after 3 hours = original amt ± net change

A in 3hr = 100 + net change

$$\int_0^3 (450.268)e^{1.125673t}\,dt$$

$u = 1.125673t$

$du = 1.125673\,dt$

$$\frac{450.268}{1.12567}\int_0^3 e^{1.125673t}\,(1.12567)dt$$

$$\frac{450.268}{1.12567}\int_0^{3.377019} e^u\,du = \frac{450.268}{1.12567}e^u\Big]_0^{3.377019} = 11313.309$$

A in 3hr = 100 + 11313 = 11413 bacteria after 3 hours

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5.5 Integration by Substitution

Rules Integration

60

$$\int_0^1 x\, e^{-x^2}\, dx$$

$u = -x^2$

$du = -2x\, dx$

Limit	x	$u=-x^2$
LL	0	0
UL	1	-1

$$-\tfrac{1}{2}\int_0^1 e^{-x^2}(-2x)\, dx$$

$$-\tfrac{1}{2}\int_0^{-1} e^u\, du$$

$$\tfrac{1}{2}\int_{-1}^0 e^u\, du = \tfrac{1}{2} e^u\Big]_{-1}^0 = \tfrac{1}{2}\left[e^0 - e^{-1}\right]$$

$$= \tfrac{1}{2}\left[1 - 1/e\right] = \frac{e-1}{2e}$$

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5.5 Integration by Substitution

Rules Integration

$$\int_0^1 x^3(2x^4+1)^2\, dx$$

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5.5 Integration by Substitution

$$\frac{1}{8}\int_0^1 8x^3(2x^4+1)^2\,dx$$

$u = 2x^4+1$

$du = 8x^3\,dx$

	$u=2x^4+1$
$x=0$	$0+1=1$
$x=1$	$2(1)+1=3$

$$= \frac{1}{8}\int_{x=0}^{x=1} u^2\,du$$

$F(b) - F(a)$

$$= \frac{1}{8}\,\frac{u^3}{3}\Big]_1^3 = \frac{1}{8}\left[\frac{3^3}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{8}\left[9 - \frac{1}{3}\right] = \frac{27-1}{3}\cdot\frac{1}{8} = \frac{26}{3\cdot 8} = \boxed{\frac{13}{12}}$$

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5.5 Integration by Substitution

Rules Integration

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$$

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5.5 Integration by Substitution

$$\frac{1}{4}\int_0^2 \frac{4x}{\sqrt{1+2x^2}}\,dx$$

$u = 1+2x^2$

$du = 4x\,dx$

	x	$u = 1+2x^2$
LL	0	$1+0=1$
UL	2	$1+8=9$

$$\frac{1}{4}\int_1^9 u^{-1/2}\,du$$

$$= \frac{1}{4}\left.\frac{u^{1/2}}{\frac{1}{2}}\right]_1^9 = \frac{1}{2}\sqrt{u}\Big]_1^9 = \frac{1}{2}\left[\sqrt{9}-\sqrt{1}\right] = \frac{1}{2}[3-1] = 1$$

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5.5 Integration by Substitution

Rules Integration

EXTRAS

Solve the diff eq: $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

$$dy = \frac{dy}{dx}dx = \frac{10x^2}{\sqrt{1+x^3}}dx$$

$u = 1+x^3$

$du = 3x^2\,dx$

$$\int dy = \int 10x^2(1+x^3)^{-1/2}dx$$

$$= \frac{10}{3}\int (1+x^3)^{-1/2}\cdot 3x^2\,dx$$

$$= \frac{10}{3}\int u^{-1/2}\,du = \frac{10}{3}\cdot\frac{u^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{10}{3}\cdot 2\sqrt{u} + c = \frac{20}{3}\sqrt{1+x^3} + c = y$$

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5.5 Integration by Substitution

Solve the diff eq: $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

$u = x^2-8x+1$
$du = (2x-8)\,dx = 2(x-4)\,dx$

$\int dy = \frac{1}{2}\int (x^2-8x+1)^{-1/2}\, 2(x-4)\,dx$

$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + c$

$= \sqrt{u} + c = \sqrt{x^2-8x+1} + c = y$

$y = \sqrt{x^2-8x+1} + c$

5.5 Integration by Substitution

Rules Integration

Solve the diff eq, and find the equations thru (2,7):

$f'(x) = -2x\sqrt{8-x^2}$

$u = 8-x^2$
$du = -2x\,dx$

$\int f'(x)\,dx = \int -2x(8-x^2)^{1/2}\,dx$

$= \int u^{1/2}\,du = \frac{2u^{3/2}}{3} + c$

$f(x) = \frac{2}{3}(8-x^2)^{3/2} + c$

$f(2) = \frac{2}{3}(8-4)^{3/2} + c = 7$

$\frac{2}{3}(\sqrt{4})^3 + c = 7 \quad = \frac{16}{3} + c$

$c = 7 - \frac{16}{3} = \frac{21-16}{3} = \frac{5}{3}$

$f(x) = \frac{2}{3}(8-x^2)^{3/2} + \frac{5}{3}$