11.8 Power Series; 11.9 Functions as Power Series

Goals:

1. Recognize Power Series as an infinite polynomial, a series in the variable *x*.
2. Power Series converge for only some values of x. We identify these values by:
   > Radius of Convergence, R, as the distance from the center value for which the series converges. eg: $|x-c|<R$
   > Interval of Convergence, IC, as the values of x for which the series converges.   eg: $c-R<x<c+R$
4. Differentiate and integrate known Power Series to find power series representations for other functions.

Study 11.8 # 1 - 5, 9 -17

Study 11.9 # 1 - 7, 11-15

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Power Series: Infinite polynomial.
Infinite series in the variable x.

If *x* is a variable, then an infinite series is a Power Series if it has the form:

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

centered at 0

$$\sum_{n=0}^{\infty} b_n (x-c)^n = b_0 + b_1 (x-c) + b_2 (x-c)^2 + \ldots$$

centered at c

$\sum_{n=0}^{\infty}$

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Examples of Power Series

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$ centered at 0

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$ centered at 0

$$\sum_{n=0}^{\infty} b_n (x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ldots$$ centered at c

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \ldots$$ centered at -1

$\sum_{n=0}^{\infty}$

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Interval of Convergence of Power Series

Find the Interval of Convergence:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$ centered at 0

Use Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$$

$\sum_{n=0}^{\infty}$

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Interval of Convergence of Power Series

Find the Interval of Convergence:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad \text{centered at } 0$$

Use Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$ converges absolutely for all x

Thus, the Radius of Convergence, R = ∞ and the interval of convergence is (-∞, ∞)

$\sum_{n=0}^{\infty}$

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Interval of Convergence of Power Series

Find the Interval of Convergence:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

Use Ratio Test:

$\sum_{n=0}^{\infty}$

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Interval of Convergence of Power Series

Find the Interval of Convergence:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

Use Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x| = |x|$$

By Ratio Test, converges when $|x| < 1$

Thus, the Radius of Convergence, R = 1 and the interval of convergence is (-1, 1)

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Interval of Convergence of Power Series

Find the Interval of Convergence:

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots$$

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Interval of Convergence of Power Series

Find the Interval of Convergence:

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots$$

Use Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}\left|(n+1)x\right|$$

$$= \begin{cases} \infty, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

By Ratio Test, diverges for all values of $x \neq 0$, and converges when $x = 0$

Thus, the Radius of Convergence, $R = 0$ and the interval of convergence, $(0,0)$

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$$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ldots$$ centered at c

For a **Power Series** centered at ***c***, **one** of the following is **true**:

1. The Series converges only at *c*. The radius of convergence is R=0.

2. The Series converges absolutely for all x. The radius of convergence is R=∞

3. There exists a real number R > 0, the radius of convergence, such that:
   the series converges absolutely for $|x-c| < R$ and diverges for $|x-c|>R$

Note: A power series can converge to either a point, an interval, or the entire real line.

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$$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ldots$$ centered at c

To determine convergence of a Power Series:

1. Use the Ratio Test to find the Radius of Convergence

2. Analyze the series at the endpoints to find the Interval of Convergence. This can be open or closed, so be careful in use of ( ) [ ], etc.

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Functions as Power Series

* $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}, \quad |x|<1$$ *

represents the function $\frac{1}{1-x}$ on its interval of convergence

* Geometric Series with a = 1 and r = x

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$ If $|r| < 1$ Converges to: $\frac{a}{1-r}$

is also a Power Series with b = 1 and c = 0

*Can consider 1/(1-x) for x=2, f(2)=-1, BUT that is not represented by the series, because the series diverges at x=2

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Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}, \quad |x|<1$$

converges to

represents the function on (-1,1)

$$f(x) = \frac{1}{1-x}$$

Can we find a power series centered at c=1
for the function f(x) = 1/x ?

$\sum_{n=0}^{\infty}$

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Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}, \quad |x|<1$$

converges to

represents the function

Can we find a power series centered at c = 1 for the function f(x) = 1/x ?

1. Rewrite the function to look like ***a/(1-r)*** and set equal to corresponding geometric series.

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} [1-x]^n$$

2. This is a geometric series with a=1 and r = x-1 and the series converges when $|1-x| < 1$ or $-1 < 1-x < 1$, $-2 < -x < 0$, $\mathbf{0 < x < 2}$

3. To write this as a power series centered at c = 1, we need the form (x-1):

$$\sum_{n=0}^{\infty} [1-x]^n = \sum_{n=0}^{\infty} [-(x-1)]^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \ldots$$

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Functions as Power Series

Find a geometric power series for the function

$f(x) = 1/(2+x)$

$\sum_{n=0}^{\infty} ar^n \qquad \frac{a}{1-r} \qquad |r|<1$

1. Rewrite the function in the form *a/(1-r)* and set equal to corresponding geometric series.

Option 1

$\frac{1}{2+x} = \frac{1}{1-(-1-x)} \qquad a = 1,\quad r = -(x+1)$

$\sum_{n=0}^{\infty} [-(x+1)]^n = \sum_{n=0}^{\infty} (-1)^n (x+1)^n \qquad |-(x+1)| < 1$

R = 1, CENTERED at x = -1

$-1 < x+1 < 1$

$1-(x+1)+(x+1)^2-(x+1)^3+\cdots \qquad IC: -2 < x < 0$

endpts $x=-2$: $\sum_{n=0}^{\infty}(-1)^n(-1)^n \to \sum_{n=0}^{\infty}(-1)^{2n}$ $1+1+1+\cdots$ diverges (n^{th} term)

$x=0$: $\sum_{n=0}^{\infty}(-1)^n$ diverges (geom) $|r|\ge 1$ ∴

$\sum_{n=0}^{\infty}$

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Functions as Power Series

Find a geometric power series for the function

$f(x) = 1/(2+x)$

1. Rewrite the function in the form *a/(1-r)* and set equal to corresponding geometric series.

Option 2 $\qquad \frac{a}{1-r}$

$f(x) = \frac{1}{2+x} = \frac{1/2}{1+(x/2)} = \frac{1/2}{1-(-x/2)}$ ∴ a=1/2 and r = -x/2

$\left|-\frac{x}{2}\right| < 1 \qquad -2 < x < 2$

$\sum_{n=0}^{\infty}(1/2)(-x/2)^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2^{n+1}} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$

2. This converges when |-x/2| < 1 or -1 < -x/2 < 1, -2 < -x < 2, **-2 < x < 2**
so the series represents the function $f(x) = 1/(2+x)$ when -2 < x < 2

3. Check endpoints: x=-2, Σ(1/2) diverges by nth term divergence.
x=+2, $\Sigma(-1)^n(1/2)$ diverges by nth term divergence. ∴ **-2 < x< 2**

R = 2, CENTERED at x = 0

$\sum_{n=0}^{\infty}$

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Functions as Power Series: Differentiation and Integration

From previous $f(x) = \frac{1}{x} = \frac{1}{1-(1-x)} = \dots = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ R = 1, $0 < x < 2$

Can we represent ***ln x*** as a **power series?**

Consider **Integration**:

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n\,dx$$

$$\ln x + c = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

let x=1, $\ln 1 + c = \sum_{n=0}^{\infty} 0 = 0 \quad \therefore \; c = 0$

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

Thus, ***ln x*** can be represented as a power series. ☺

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Functions as Power Series: Differentiation and Integration

From previous $f(x) = \frac{1}{x} = \frac{1}{1-(1-x)} = \dots = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ $0 < x < 2$

Integration: $\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ $\int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n\,dx$

$\ln x + c = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$, c = 0

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

Find: R and IC, Use Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+2}/(n+2)}{(x-1)^{n+1}/(n+1)}\right| = \lim_{n\to\infty} \left|\frac{(x-1)(n+1)}{(n+2)}\right| = |x-1|$$

R=1, as before integration, but need to check endpoints $|x-1| < 1$ $0 < x < 2$

let x=0: Limit Comparison Test

$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{(n+1)} = 1$

since 1/n diverges, so does 1/(n+1) by Limit Comparison Test

let x=2 $\lim_{n\to\infty} \frac{(-1)^n}{n+1} = 0$

alternating series

$1/(n+2) < 1/(n+1)$

converges at x = 2

IC: $0 < x \le 2$

gained a pt.

Note: I disagree with online calculator for x=0. It assumes that there are alternating signs, but the signs are not alternating bec. 2n+1 is always odd and $(-1)^{2n+1}$ is always <0

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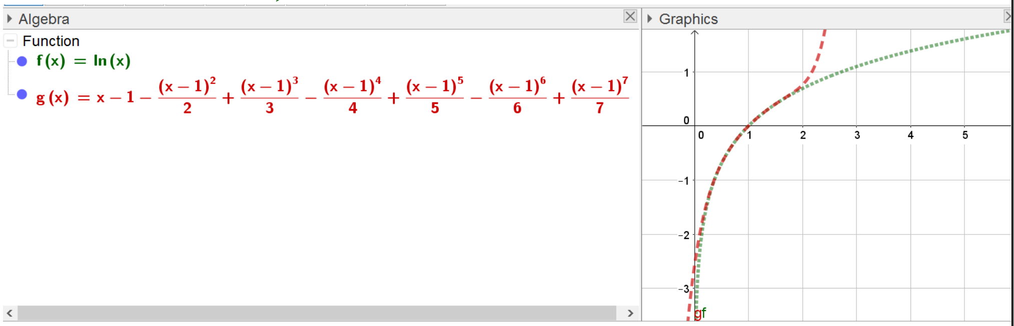

$$\ln x = \sum_{n=0} (-1)^n \frac{(x-1)^{n+1}}{n+1} = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

For R and IC, Use Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+2}/(n+2)}{(x-1)^{n+1}/(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(x-1)(n+1)}{(n+2)}\right| = |x-1|$$

R=1, as before integration, but need to check endpoints $|x-1| < 1$ $0 < x < 2$

let x=0:

$$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{(n+1)} = 1$$

since 1/n diverges, so does 1/(n+1) by Limit Comparison Test

let x=2 $\lim_{n\to\infty} \frac{(-1)^n}{n+1} = 0$

alternating series

$1/(n+2) < 1/(n+1)$

converges at x = 2

IC: $0 < x \le 2$

gained a pt.

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Functions as Power Series

Find a power series representation for

$f(x) = 1/(1-x)^2$

1. Recognize that $1/(1-x)^2$ is the **derivative** of $1/(1-x)$.

2. Start with $1/(1-x)$ and rewrite the function in the form ***a/(1-r)*** and set equal to corresponding geometric series.

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Functions as Power Series

Find a power series representation for
$f(x) = 1/(1-x)^2$

1. Recognize that $1/(1-x)^2$ is the **derivative** of $1/(1-x)$.

2. Start with $1/(1-x)$ and rewrite the function in the form ***a/(1-r)*** and set equal to corresponding geometric series.

$g(x) = \frac{1}{1-x}$ already in form needed a=1, r = x, and $\sum_{n=0}^{\infty} x^n$ |x|<1,R=1

$$g'(x) = f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1}$$

3. For R and IC, this is a **geometric** power series which converges when $|x| < 1$ and diverges for $|x| \geq 1$, **so R = 1** (same as for g(x)), **and IC: (-1,1)**

or $\sum_{n=0}^{\infty}(n+1)x^n$

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$$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ \ldots$$ centered at c

May lose endpoints when you differentiate a power series and gain endpoints when you integrate.

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